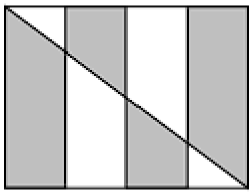
October Math Gems

Problem of the week 27

§1 Problems

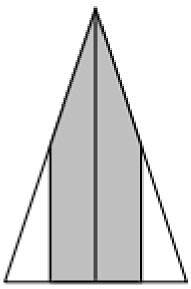
Problem 1.1.

Below is a diagram showing a 6×8 rectangle divided into four 6×2 rectangles and one diagonal line. Find the total perimeter of the four shaded trapezoids.



Problem 1.2.

An isosceles triangle has a base with length 12 and the altitude to the base has length 18. Find the area of the region of points inside the triangle that are a distance of at most 3 from that altitude.



Problem 1.3.

Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Problem 1.4.

In quadrilateral $ABCD$, let $AB = 7$, $BC = 11$, $CD = 3$, $DA = 9$, $\angle BAD = \angle BCD = 90^\circ$, and diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. The ratio $\frac{BE}{DE} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 1.5.

Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Problem 1.6.

Let x be a real number such that $\left(\sqrt{6}\right)^x - 3^x = 2^{x-2}$. Evaluate $\dfrac{4^{x+1}}{9^{x-1}}$.

Problem 1.7.

Let $ABCD$ be a convex quadrilateral inscribed in a circle with $AC = 7$, $AB = 3$, $CD = 5$, and $AD - BC = 3$. Then $BD = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 1.8.

The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 1.9.

Find the maximum possible value obtainable by inserting a single set of parentheses into the expression $1 + 2 \times 3 + 4 \times 5 + 6$.

Problem 1.10.

Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

Problem 1.11.

In $\triangle ABC$ let point D be the foot of the altitude from A to $\overline{BC}$. Suppose that $\angle A = 90^\circ$, $AB - AC = 5$, and $BD - CD = 7$. Find the area of $\triangle ABC$.

Problem 1.12.

A rectangle with width 30 inches has the property that all points in the rectangle are within 12 inches of at least one of the diagonals of the rectangle. Find the maximum possible length for the rectangle in inches.

Problem 1.13.

Let a and b be positive integers satisfying $3a < b$ and $a^2 + ab + b^2 = (b+3)^2 + 27$. Find the minimum possible value of $a + b$.

Problem 1.14.

Find the positive integer n such that a convex polygon with $3n + 2$ sides has 61.5 percent fewer diagonals than a convex polygon with $5n - 2$ sides.

Problem 1.15.

Find the number of divisors of 20^{22} that are perfect squares.

Problem 1.16.

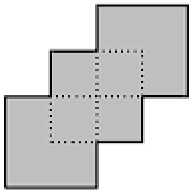
The value of

$$\left(1 - \frac{1}{2^2 - 1}\right)\left(1 - \frac{1}{2^3 - 1}\right)\left(1 - \frac{1}{2^4 - 1}\right)\cdots\left(1 - \frac{1}{2^{29} - 1}\right)$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $2m - n$.

Problem 1.17.

The 12-sided polygon below was created by placing three 3×3 squares with their sides parallel so that vertices of two of the squares are at the center of the third square. Find the perimeter of this 12-sided polygon.



Problem 1.18.

A rectangular wooden block has a square top and bottom, its volume is 576, and the surface area of its vertical sides is 384. Find the sum of the lengths of all twelve of the edges of the block.

Problem 1.19.

Find the value of x where the graph of

$$y = \log_3\left(\sqrt{x^2+729}+x\right) - 2\log_3\left(\sqrt{x^2+729}-x\right)$$

crosses the x-axis.

Problem 1.20.

A semicircle has diameter $\overline{AB}$ with $AB = 100$. Points C and D lie on the semicircle such that $AC = 28$ and $BD = 60$. Find CD.
